4.3 Areas and Volumes

Problems Worksheet

$$\pi \int_a^b y^2 \, dx$$
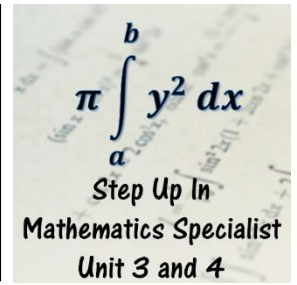
1. Consider the functions $f(x) = 7 - x^2$, $g(x) = x + 5$ and $h(x) = |x| + 1$.
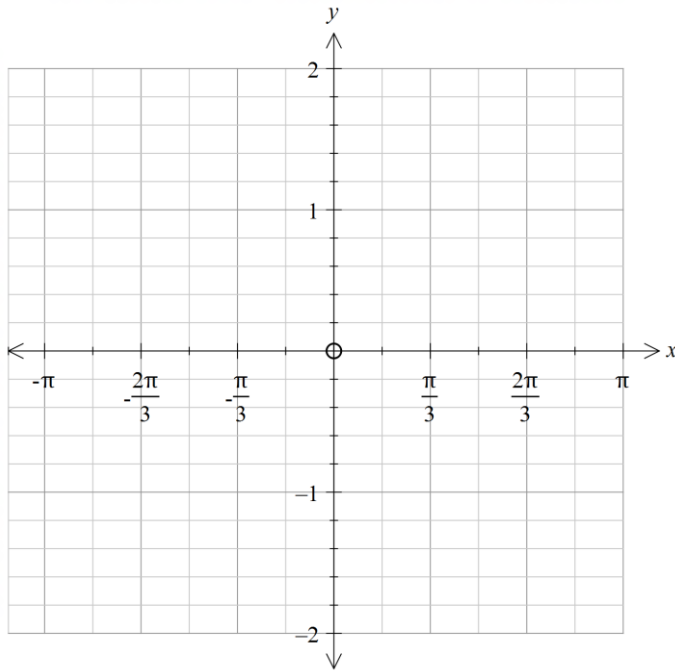
   a. Determine exactly the area between $f(x)$ and the x axis.

   b. Determine exactly the area bound by $f(x)$ and $g(x)$.

   c. Determine exactly the area bound by $f(x)$ and $h(x)$.

2. Consider the functions $m(x) = \sin x$ and $n(x) = 0.5x$.

a. Sketch $m(x)$ and $n(x)$ on the axes below. Label the points of intersection accurate to 3dp.

b. Determine the value of $\int_0^{1.895} (m(x) - n(x))\, dx$ to 3dp and describe the physical meaning of this quantity.

c. Determine the value of $\int_0^{1.895} (n(x) - m(x))\, dx$ to 3dp and describe the physical meaning of this quantity.

d. Determine the value of $\int_{-1.895}^{1.895} (m(x) - n(x))\, dx$ to 3dp and describe the physical meaning of this quantity.

e. Write two separate integrals which, when evaluated, would give the area enclosed between $m(x)$ and $n(x)$.

3. Determine the volumes of the following revolutions exactly.

   a. The area between $y = \sqrt{\sin x}$ and the x axis, for $0 \leq x \leq \pi$, about the x axis.

   b. The area between $y = -(x-3)^2 + 4$ and the x axis, for $1 \leq x \leq 5$, about the y axis. A ClassPad may be used for the integration.

4. Determine the volume of the solid formed when the area between $y = x^2$, $x = 2$ and the x axis is rotated:

   a. About the x axis .

   b. About the y axis.

5. Gabriel's Horn is the name given to the geometric figure formed by rotating the area between $y = \frac{1}{x}$ and the x axis around the x axis, for $x > 0$. It can be shown mathematically that this 3D shape will have a surface area approaching ∞ as x continues to increase, however this calculation is outside the scope of this course.

    a. Write an integral to calculate the volume of the solid formed as described above, with $1 \le x \le k$.

    b. Evaluate this integral for $k \to \infty$.

    c. Describe the apparent paradox between the question and your result to b.

6. An ornamental bowl is to be manufactured such that its outside shape is defined by the curve $x = \sqrt{y} + 1 + \sin 0.5y$. The outer height of the bowl will be 2π units and it will be of uniform thickness 0.5 units. Write appropriate integrals, and then with the aid of a ClassPad calculate the following to 2dp:

    a. The volume of material required to manufacture the bowl.

    b. The internal volume of the bowl.

7. Prove the following formulae using the concept of volumes of revolution.

    a. The volume of a cylinder, base radius r and height h, is given by $V = \pi r^2 h$.

    b. The volume of a cone, base radius r and height h is given by $V = \frac{1}{3}\pi r^2 h$.

    c. The volume of a sphere, radius r, is given by $V = \frac{4}{3}\pi r^3$.

8. The area between $y = 9 - x^2$ and $y = x + 3$ for $x \geq 0$ is rotated about the line $y = 1$. Determine the volume of revolution.

9. Write an expression for the volume of a doughnut which has a cross-section that is circular. Let R be the distance measured from the centre of the hole to the centre of the dough, and r is the radius (thickness) of the circular ring itself. Simplify the integral(s) which result, however do not evaluate your answer.